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Technical Note

Effect of viscous dissipation and pressure stress work in natural convection along a vertical isothermal plate. New results

A. Pantokratoras *

School of Engineering, Democritus University of Thrace, 67100 Xanthi, Greece

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Abstract

The steady laminar boundary layer flow along a vertical stationary isothermal plate is studied taking into account the viscous dissipation and pressure stress work of the fluid. The results are obtained with the direct numerical solution of the boundary layer equations without any approximation. It was found that the variation of wall heat transfer and wall shear stress along the plate is quite different compared to that given by the approximate perturbation method.

Keywords: Isothermal plate; Natural convection; Viscous dissipation; Pressure work

1. Introduction

The problem of natural convection along a vertical isothermal plate is a classical problem of fluid mechanics that has been solved with the similarity method 50 years before [9]. In this work the viscous dissipation term in the energy equation has been omitted (second term in the RHS of Eq. (3) of the present paper). Ostrach [10] has shown that viscous dissipation plays an important role in natural convection in vertical channels. Gebhart [2] was the first who studied the problem of laminar natural convection along a vertical heated plate taking into account the viscous dissipation. When the viscous dissipation term is added the problem does not admit similarity solution. Ostrach [10] and Gebhart [2] found that the nondimensional parameter $g\beta l/c_p$ (g = acceleration due to gravity, β = coefficient of thermal expansion, l = length scale of the problem, c_p = specific heat under constant pressure) determined the influence of viscous dissipation and it has been called the dissipation number. Sometimes this number is referred in the literature as the Gebhart number [8,15]. The solution obtained by Gebhart [2] was based on the perturbation method by retaining only one perturbation function for velocity and two perturbation functions for temperature. The rest of the perturbation functions were omitted. Gebhart and Mollendorf [3] have shown that the problem of natural convection over a vertical plate with viscous dissipation admits similarity solution only when the plate temperature varies in exponential law. Kuiken [6] gave a similarity solution for natural convection along a vertical plate with linearly varying temperature ignoring the viscous dissipation term and taking into account the pressure work term in the energy equation (last term in Eq. (3) of the present paper). Ackroyd [1] was the first who treated the problem of natural convection along a heated vertical plate taking into account both the viscous dissipation and the pressure work in the energy equation. He proved that, for this problem, the pressure work effect is more important than that of viscous dissipation. The solution was obtained using the perturbation method by retaining two perturbation functions for both velocity and temperature. The rest of the perturbation functions were omitted. He presented results for Pr numbers 0.72 and 7. Joshi and Gebhart [5]

* Tel.: +30-541-079618.

E-mail address: apantokr@civil.duth.gr (A. Pantokratoras).

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treated, with the perturbation method, the problem of natural convection over a vertical isothermal plate taking into account both the viscous dissipation and the pressure work in the energy equation using two non-dimensional numbers. The first was the old dissipation number multiplied by 4 ($\varepsilon = 4g\beta x/c_p$) and the second was a new number defined as $\lambda = \varepsilon T_f/\Delta T$ where T_f is the mean temperature between the plate and the ambient fluid (film temperature) and ΔT is the temperature difference between the plate and the ambient fluid. They presented results for *Pr* numbers 0.733, 10 and 100. Mahajan and Gebhart [7], after conducting an order of magnitude analysis, have shown that the viscous dissipation effect is smaller than the pressure effect for all values of *Pr* number.

From the above analysis it is clear that for the problem of natural convection along a vertical isothermal plate with viscous dissipation and pressure work only approximate solutions exist until now. The objective of the present paper is to obtain results without any approximation. As will be shown later the differences of the two methods are very large.

It should be noted here that the inclusion of viscous dissipation and pressure work terms in the energy equation, except of the theoretical interesting, has applications in glaciology, in granular material, in the infall of molten iron during gravitational differentiation of terrestrial planets, in the interaction between the crust and mantle during continental convergence and in the separation of oceanic crust from the descending oceanic lithosphere. Much work on these fields has been done by Yuen and his co-workers [13,14,16].

2. The mathematical model

Consider laminar free convection along a vertical plate placed in a calm environment with u and v denoting respectively the velocity components in the x and y direction, where x is vertically upwards and y is the coordinate perpendicular to x. For steady, two-dimensional flow the boundary layer equations including viscous dissipation and pressure work are

continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \beta g(T - T_a) \tag{2}$$

energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\beta T}{\rho c_p}u\frac{dp}{dx} \tag{3}$$

where T is the fluid temperature, ν is the kinematic viscosity, β is the fluid expansion coefficient, α is the thermal diffusivity, c_p is the specific heat at constant pressure, ρ is the fluid density and p is the pressure. The last two terms in the energy equation are the viscous dissipation and the pressure stress work respectively. The fluid pressure consists of the hydrostatic and motion pressure:

$$p = p_h + p_m \tag{4}$$

The motion pressure is considered small compared to hydrostatic pressure and is ignored [5]. For the hydrostatic pressure we have

$$\frac{dp_h}{dx} = -\rho g \tag{5}$$

Under these conditions the energy equation takes the form

energy equation:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 - \frac{\beta T}{c_p}ug \tag{6}$$

The viscous dissipation term is always positive and represents a source of heat due to friction between the fluid particles. A variety of expressions are used in the literature for this term like viscous heating, shear-stress heating and viscous work. The pressure work is the work required to push fluid into or out of a control volume. When fluid cross a control surface and enters the control volume, it must push back the fluid that is already inside the control volume. Since that fluid has a pressure, the entering fluid must do work to move it. For example, rising air expands because as it rises there is less atmospheric pressure compressing it and as it expands becomes cooler. This phenomenon is called adiabatic cooling. The opposite happens when air sinks. Sinking air is compressed and becomes warmer. This phenomenon is called adiabatic heating. For the pressure work term the expressions adiabatic temperature gradient, adiabatic gradient and adiabatic heating or adiabatic cooling are used. The pressure work term in Eq. (6) is negative for rising fluid according to the above analysis (adiabatic cooling). The viscous dissipation tends to rise the fluid temperature while the pressure work tends to lower its temperature in the upward flow examined here.

For the isothermal plate the boundary conditions are as follows:

$$\begin{aligned} &\text{at } y = 0, && u = v = 0,\ T = T_w \\ &\text{as } y \to \infty, && u = 0,\ T = T_a \end{aligned}$$

where T_w is the plate temperature and T_a is the ambient fluid temperature.

Eqs. (1), (2) and (6) form a parabolic system and were solved directly, without any transformation, by a method described by Patankar [12]. The finite difference method is used with primitive variables x, y and a space marching procedure is used in x direction with an expanding grid. A detailed description of the solution procedure may be found in Pantokratoras [11].

3. Results and discussion

As was mentioned before the classical problem of natural convection along a vertical isothermal plate without viscous dissipation and pressure work admits similarity solution. The most important quantities for this problem are the wall heat transfer and the wall shear stress defined as

$$\theta'(0) = \frac{x}{T_w - T_a}\left(\frac{Gr_x}{4}\right)^{-1/4}\left[\frac{\partial T}{\partial y}\right]_{y=0} \tag{7}$$

$$f''(0) = \frac{x^2}{v\sqrt{2}}(Gr_x)^{-3/4}\left[\frac{\partial u}{\partial y}\right]_{y=0} \tag{8}$$

where θ is the dimensionless temperature $(T - T_a)/(T_w - T_a)$ and f is the dimensionless stream function for which the following equation is valid:

$$f' = \frac{ux}{2v}(Gr_x)^{-1/2} \tag{9}$$

The local Grashof number is defined as

$$Gr_x = \frac{g\beta x^3}{v^2}(T_w - T_a) \tag{10}$$

In Eqs. (7)–(9) the prime represents differentiation with respect to similarity variable η defined as

$$\eta = \frac{y}{x}\left(\frac{Gr_x}{4}\right)^{1/4} \tag{11}$$

Although the problem with viscous dissipation and pressure work does not admit similarity solution our results have been produced using the above similarity quantities because it is easy to see the divergence of the nonsimilar solution from the similar one which is used as the basis for this problem.

In order to test the accuracy of the present method, results were compared with those available in the literature. In the work of Gebhart [4] a complete table is given with data concerning the transport quantities of free convection adjacent to a vertical isothermal plate without viscous dissipation and pressure work. This table was prepared by Krishnamurthy. For example the quantities $\theta'(0)$ and $f''(0)$ for $Pr = 6.7$ calculated with the present method were -1.0431 and 0.4509 and the corresponding values by Krishnamurthy were -1.0408 and 0.4548. The comparison is considered satisfactory and this happens for all Pr numbers.

In Fig. 1 the wall heat transfer is shown as function of the distance from the plate leading edge for $Pr = 10$ and different values of the quantities ε/x and λ/x. The units of

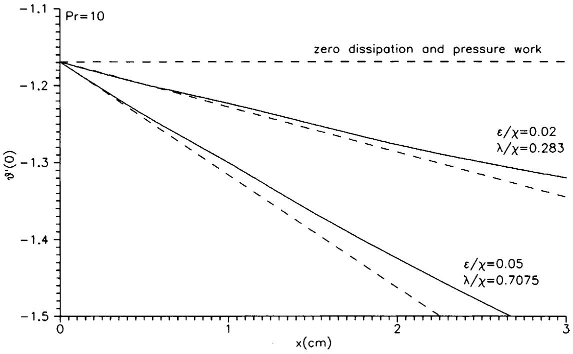

Fig. 1. Wall heat transfer for an isothermal plate and $Pr = 10$. The dashed horizontal line corresponds to $\theta'(0)$ without viscous dissipation and pressure work. Solid line curves correspond to $\theta'(0)$ of the present work and straight dashed lines to work of Joshi and Gebhart [5].

these quantities are cm^{-1}. It should be noted here that the boundary layer theory is inapplicable at the plate leading edge. The solution procedure starts with a boundary layer with zero thickness and the boundary layer takes its complete form at some downstream position from the leading edge. However, the results are shown starting at $x = 0$ only for sake of good presentation of figures. This remark is valid for all figures. The dashed horizontal line in Fig. 1 corresponds to wall heat transfer of the classical problem of free convection along a vertical isothermal plate without viscous dissipation and pressure work. This value is -1.16898 [5]. Solid line curves correspond to $\theta'(0)$ of the present work and dashed line curves to the work of Joshi and Gebhart [5]. As was mentioned before Joshi and Gebhart produced their results with a perturbation method. They retained only first order terms while higher order terms were omitted. In Fig. 2 the wall shear stress is shown as function of the distance from the leading edge for the same conditions of Fig. 1. Now the dashed horizontal line corresponds to wall shear stress of the classical problem of free convection along a vertical isothermal plate with zero viscous dissipation and pressure work. This value is 0.4192 [5]. From Figs. 1 and 2 it is seen that the variation of $\theta'(0)$ and $f''(0)$ with x is linear according to Joshi and Gebhart and nonlinear according to the present work. The two methods give identical results only for small values of x. It is seen that as x increases the divergence increases, too.

The next Pr number for which Joshi and Gebhart [5] presented results is $Pr = 100$. In Figs. 3 and 4 the wall heat transfer and the wall shear stress are shown as functions of the distance from the plate leading edge. The trends of Fig. 4, concerning wall shear stress, are the same with those of Fig. 2. The interesting point is in Fig. 3 where we see that the wall heat transfer increases with increasing distance x according to Joshi and Gebhart [5]. These predictions are in agreement with the values given

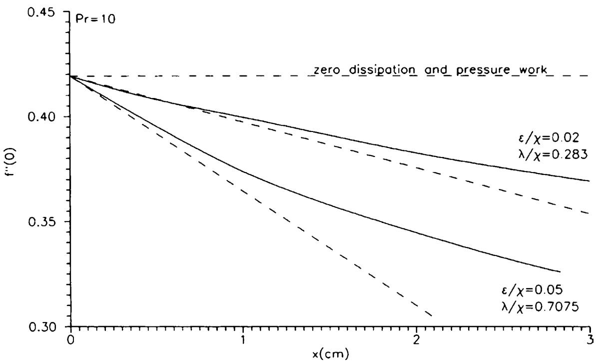

Fig. 2. Wall shear stress for an isothermal plate and $Pr = 10$. The dashed horizontal line corresponds to $f''(0)$ without viscous dissipation and pressure work. Solid line curves correspond to $f''(0)$ of the present work and straight dashed lines to work of Joshi and Gebhart [5].

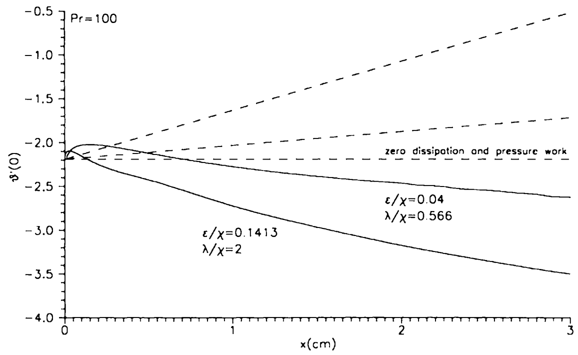

Fig. 3. Wall heat transfer for an isothermal plate and $Pr = 100$. The dashed horizontal line corresponds to $\theta'(0)$ without viscous dissipation and pressure work. Solid line curves correspond to $\theta'(0)$ of the present work and straight dashed lines to work of Joshi and Gebhart [5].

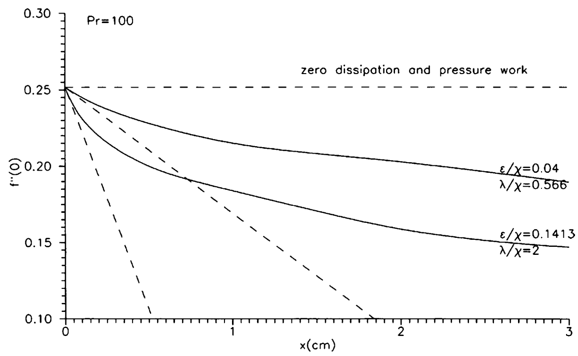

Fig. 4. Wall shear stress for an isothermal plate and $Pr = 100$. The dashed horizontal line corresponds to $f''(0)$ without viscous dissipation and pressure work. Solid line curves correspond to $f''(0)$ of the present work and straight dashed lines to work of Joshi and Gebhart [5].

in Table 1 of their work. They emphasized this reversal of the wall heat transfer and tried to give an explanation. The wall heat transfer of the present work, although increases a little near the plate edge, follows the general trend which is the reduction with increasing x.

Results have been also produced for Pr numbers 100, 1000 and 10,000 but are not shown. The general behavior of wall heat transfer and wall shear stress is the same with that in the previous figures. Both quantities decrease as the distance from the plate leading edge increases. The explanation is obvious. As the fluid rises the hydrostatic pressure decreases along x, the fluid expands and becomes cooler (pressure work). At the same time heat is produced due to friction between the moving fluid particles (viscous heating) but this effect is small compared to adiabatic cooling as Mahajan and Gebhart [7] have shown. The continuous fluid cooling causes a continuous reduction of wall heat transfer and wall shear stress.

Another interesting phenomenon which appears in this flow is the following. The fluid temperature at the outer part of the boundary layer becomes smaller than the ambient temperature due to continuous cooling of the fluid. As this negative temperature difference is small the entire fluid moves upward. As the fluid becomes more and more cooler the fluid at this region starts to move downwards while the fluid near the plate moves upwards. This means that at some distance from the leading edge a bidirectional flow appears. The parabolic solution procedure used in the present work cannot simulate this bidirectional flow and breaks down. For the simulation of this phenomenon an elliptic solution procedure is needed and this is a proposal for future research.

4. Conclusions

The foregoing results are the first exact calculations of the effect of viscous dissipation and pressure work on the classical problem of natural convection along a vertical isothermal plate and can be summarized as follows:

1. The variation of wall heat transfer and wall shear stress is nonlinear along the plate in contrary to the variation given by the perturbation method which is linear.
2. The two methods give identical results only for small values of x. As x increases the divergence increases, too.
3. At some downstream position from the plate leading edge the flow becomes bidirectional with the fluid near the plate moving upwards and the fluid away from the plate moving downwards.
4. The wall heat transfer and wall shear stress decrease along the plate due to continuous cooling of the fluid.

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